International Journal of Theoretical Physics, Vol. 33, No. 8, 1994

Quantum Deformation of Superalgebra

Won-Sang Chung[1]

Received November 19, 1993

In this paper we discuss two types of q-deformations of superalgebra.

A new type of algebraic structure called a quantum group has been created by Jimbo (1985, 1986) and Drinfel'd (1986). One of the simplest examples of a quantum group is $su_q(2)$ and its realization has been obtained by Macfarlane (1989) and Biedenharn (1989). More recently some physicists have considered a q-analog of the supersymmetric oscillator including the q-creation and q-annihilation operators (Parthasarathy and Viswanathan, 1991; Viswanathan *et al.*, 1992; Chaichin and Kulish, 1990; Lukierski and Nowicki, 1992).

In this paper we will discuss two types of supersymmetric versions of $su_q(2)$ algebra, known as $osp_q(1/2)$ algebra. The $osp_q(1/2)$ algebra can be represented by five generators, which we denote as $J_\pm$, $V_\pm$ and J_3.

Our first assumption is

$$V_+^2 = f(q)J_+ \tag{1}$$

$$V_-^2 = g(q)J_- \tag{2}$$

$$[J_3, V_\pm] = \pm\tfrac{1}{2}V_\pm \tag{3}$$

where $f(q)$ and $g(q)$ are functions of q and will be fixed later. From the above relations the following commutation relation holds automatically:

$$[J_3, J_\pm] = \pm J_\pm \tag{4}$$

Our second assumption is to regard the anticommutator between V_+ and V_- as an arbitrary function of J_3, $f(J_3)$.

[1]Theory Group, Department of Physics, College of Natural Sciences, Gyeongsang National University, Jinju, 660-701, Korea.

0020-7748/94/0800-1611$07.00/0

In what follows we have

$$\{V_+, V_-\} = F(J_3) \tag{5}$$

$$[J_+, V_-] = \frac{1}{f(q)}\left[F\left(J_3 - \frac{1}{2}\right) - F(J_3)\right]V_+ \tag{6}$$

$$[J_-, V_+] = \frac{1}{g(q)}\left[F\left(J_3 + \frac{1}{2}\right) - F(J_3)\right]V_- \tag{7}$$

$$[J_+, J_-] = \frac{1}{f(q)g(q)}\left\{\left[F\left(J_3 - \frac{1}{2}\right) - F(J_3)\right]V_+ V_- + \left[F(J_3) - F\left(J_3 + \frac{1}{2}\right)\right]V_- V_+\right\} \tag{8}$$

If we assume that $V_+ V_-$ and $V_- V_+$ have the forms

$$V_+ V_- = H(J_3) \tag{9}$$

$$V_- V_+ = K(J_3) \tag{10}$$

then $F(J_3)$ is given by

$$F(J_3) = H(J_3) + K(J_3) \tag{11}$$

In this case the commutator between J_+ and J_- is

$$\begin{aligned}[J_+, J_-] &= \frac{1}{f(q)g(q)}\left\{\left[F\left(J_3 - \frac{1}{2}\right) - F(J_3)\right]H(J_3) + \left[F(J_3) - F\left(J_3 + \frac{1}{2}\right)\right]\right\}K(J_3) \\ &= \frac{1}{f(q)g(q)}\left\{H\left(J_3 - \frac{1}{2}\right)H(J_3) - K\left(J_3 + \frac{1}{2}\right)K(J_3) \right. \\ &\quad \left. + K\left(J_3 - \frac{1}{2}\right)H(J_3) - H\left(J_3 + \frac{1}{2}\right)K(J_3) + K(J_3)^2 - H(J_3)^2\right\}\end{aligned} \tag{12}$$

If we set

$$H(x) = [ax + b] \tag{13}$$

$$K(x) = [cx + d] \tag{14}$$

we can determine a, b, c, and d from the classical version of $osp_q(1/2)$ algebra, which is obtained in the limit $q \to 1$. Here

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}$$

When $q = 1$, equations (13) and (14) become

$$H(x) = ax + b \tag{15}$$

$$K(x) = cx + d \tag{16}$$

Inserting equations (15) and (16) into equation (12) gives

$$[J_+, J_-] = -\frac{(a+c)}{2f(1)g(1)}\{(a+c)J_3 + (b+d)\} \tag{17}$$

Since we know that the right-hand side of equation (17) is $2J_3$ from the classical $osp(1/2)$ algebra, the second term in brackets of equation (17) should vanish, which means $a + c = 0$ or $b + d = 0$. In the former case, we have $[J_+, J_-] = 0$ and we meet with a contradiction. Therefore we discard the first possibility and choose $d = -b$. Therefore a and c should satisfy

$$-\frac{(a+c)^2}{2f(1)g(1)} = 2 \tag{18}$$

If we choose the case that $a = c$, then b remains unfixed. Then, computing the commutator between J_+ and V_-, we have

$$[J_+, V_-] = -\frac{(q^{b/2} + q^{-b/2})(q^{a/8} - q^{-a/8})(q^{(a/2)(J_3 - 1/4)} + q^{-(a/2)(J_3 - 1/4)})}{f(q)(q^{1/2} - q^{-1/2})} V_+ \tag{19}$$

If we define the three types of q-number

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \tag{20}$$

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{21}$$

$$[x]_+ = \frac{q^{x/2} + q^{-x/2}}{q^{1/2} + q^{-1/2}} \tag{22}$$

then equation (19) reduces to

$$[J_+, V_-] = -\frac{(q^{1/2} + q^{-1/2})^2}{f(q)}\left[\frac{a}{4}\right][b]_+\left[a\left(J_3 - \frac{1}{4}\right)\right]_+ V_+ \tag{23}$$

When q goes to 1, equation (23) reduces to

$$[J_+, V_-] = -\frac{1}{2f(1)} V_+ \tag{24}$$

Then we must demand that $f(1) = 1/2$. The above three types of quantum number satisfy the property

$$[x][x]_+ = [x]_q \tag{25}$$

Similarly we find

$$[J_-, V_+] = \frac{(q^{1/2}+q^{-1/2})^2}{g(q)} \left[\frac{a}{4}\right][b]_+ \left[a\left(J_3+\frac{1}{4}\right)\right]_+ V_- \tag{26}$$

We also should demand that $g(1) = -1/2$.
Then we have

$$\begin{aligned}[J_+, J_-] &= -\frac{1}{f(q)g(q)}(q^{1/2}+q^{-1/2})^2\left[\frac{a}{4}\right][b]_+ \\ &\quad \times \left\{\left[a\left(J_3-\frac{1}{4}\right)\right]_+ [J_3+b] + \left[a\left(J_3+\frac{1}{4}\right)\right]_+ [J_3-b]\right\} \\ &= -\frac{1}{f(q)g(q)}(q^{1/2}+q^{-1/2})^2\left[\frac{a}{4}\right][b]_+\left[b-\frac{a}{4}\right]_+ [J_3]\end{aligned} \tag{27}$$

When q goes to 1, equation (27) reduces to

$$[J_+, J_-] = 4aJ_3 \tag{28}$$

Therefore we obtain $a = 1/2$, which is consistent with equation (18). If we choose

$$f(q) = \frac{1}{q^{1/2}+q^{-1/2}}$$

$$g(q) = -\frac{1}{q^{1/2}+q^{-1/2}}$$

then we arrive at the following result:

$$[J_+, J_-] = (q^{1/2}+q^{-1/2})^4\left[\frac{1}{8}\right][b]_+\left[b-\frac{1}{8}\right]_+ [J_3] \tag{29}$$

$$[J_+, V_-] = -(q^{1/2}+q^{-1/2})^3\left[\frac{1}{8}\right][b]_+\left[\frac{1}{2}\left(J_3-\frac{1}{4}\right)\right]_+ V_+ \tag{30}$$

$$[J_-, V_+] = (q^{1/2}+q^{-1/2})^3\left[\frac{1}{8}\right][b]_+\left[\frac{1}{2}\left(J_3+\frac{1}{4}\right)\right]_+ V_- \tag{31}$$

If we choose $F(J_3)$ as the function different from equation (11), we can obtain another type of q-deformation for $osp(1/2)$ algebra. For example, we set

$$F(J_3) = [J_3]_q \tag{32}$$

In this case we have

$$[J_+, V_-] = -(q^{1/4} + q^{-1/4})\left[\frac{1}{2}\right]\left[2J_3 - \frac{1}{2}\right]_+ V_+ \tag{33}$$

$$[J_-, V_+] = (q^{1/4} + q^{-1/4})\left[\frac{1}{2}\right]\left[2J_3 + \frac{1}{2}\right]_+ V_- \tag{34}$$

$$[J_+, J_-] = -\left[\frac{1}{2}\right]_q^2 + \left[\frac{1}{2}\right]_+ \left[2J_3 + \frac{1}{4}\right]_q - (q - q^{-1})[J_3]_q V_+ V_- \tag{35}$$

where we choose $f(q)$ and $g(q)$ as

$$f(q) = \frac{1}{q^{1/4} + q^{-1/4}}$$

$$g(q) = -\frac{1}{q^{1/4} + q^{-1/4}}$$

In this paper we have discussed two types of q-deformations of $osp(1/2)$ algebra. We think that other types of q-deformation of superalgebra may exist and that this method will be applicable to the deformation of other types of superalgebra. We hope that these problems and related topics will be clarified in the near future.

ACKNOWLEDGMENT

This paper was supported by Non Directed Research Fund, Korea Research Foundation, 1993.

REFERENCES

Biedenharn, L. (1989). *Journal of Physics A*, **22**, L873.
Chaichan, M., and Kulish, P. (1990). *Physics Letters B*, **234**, 72.
Drinfel'd, V. (1986). In *Proceedings of the International Congress of Mathematicians* (Berkeley, California).
Jimbo, M. (1985). *Letters in Mathematical Physics*, **10**, 63.
Jimbo, M. (1986). *Letters in Mathematical Physics*, **11**, 247.
Lukierski, J., and Nowicki, A. (1992). *Journal of Physics A*, **25**, L161.
Macfarlane, A. (1989). *Journal of Physics A*, **22**, 4581.
Parthasarathy, R., and Viswanathan, K. (1991). *Journal of Physics A*, **24**, 613.
Viswanathan, K., Parthasarathy, R., and Jagannathan, R. (1992). *Journal of Physics A*, **25**, L335.